12. Hamel, G., Die Lagrange-Eulershe Gleichungen der Mechanik. Z. Math. und Phys., Bd. 50, 1904.
13. Chaplygin, S. A., On a motion of a heavy solid of revolution on a horizontal plane. Tr. Otd. fiz. nauk o-va liubitelei estestvoznaniia, Vol. 9, 1897.
14. Voronets, P. V., On the equations of motion of nonholonomic systems. Matem. sb. Vol. 22, № 4, 1901.
15. Neimark, Iu. I. and Fufaev, N. A., Dynamics of Nonholonomic Systems. Moscow, "Nauka", 1967.

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# ON A VERSION OF THE NONLINEAR DYNAMICAL THEORY OF THIN MULTILAYERED SHELLS

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A version of the geometrically nonlinear theory of elastic multilayered shells subjected to a nonconservative load is proposed. Transverse shear strains in the layers and strains in the direction of the normal to the middle surface are taken into account. As a rule, a description of the nonstationary dynamical processes associated with shell buckling can be performed on the basis of a geometrically nonlinear theory [1]. The behavior of multilayered plates and shells under large deflections has been examined in [2—5]. A variational formulation, which is valid for conservative loads acting on a shell, is used in [5] to derive the geometrically nonlinear equations. The variational principle is formulated in this paper in a form also applicable in the case of no potential of the external forces. One of the advantages of the approach developed here as compared with the results of [5] is the additional possibility of describing the local dynamical buckling of the shell in modes associated with the change in its thickness.

**1. Variational principle for a three-dimensional body.** The variational principle of elasticity theory for a three-dimensional body under large displacements is written as follows:

$$\delta J_0 = \delta \int_{t_0}^{t_1} \Big( \int_V \Big\{ -\frac{1}{2} E^{ikjl} \varepsilon_{ik}\varepsilon_{jl} + \sigma^{ik}\Big[\varepsilon_{ik} - \frac{1}{2}(\eta_{ik} + \eta_{ki} + \eta_{i}^{\cdot j}\eta_{kj})\Big] + \tag{1.1}$$

$$\theta^{ik}(\eta_{ik} - \overline{\nabla}_i u_k) + \frac{1}{2}\rho \frac{\partial u_i}{\partial t}\frac{\partial u^i}{\partial t}\Big\} dV + \int_{S_1} P^i u_i dS +$$

$$\int_{S_2} (u_k - U_k)\,\theta^{ik} n_i dS \Big) = 0, \quad i,\ k = 1, 2, 3$$

$$\delta P^i = 0 \tag{1.2}$$

Here $E^{ikjl}$ is the elasticity tensor, $\rho$ is the material density, $u_i$ are the displacement vector components in the metric $g_{ik}$ of an undeformed body, $V$ is the volume of the undeformed body, $S_1$ is the part of the body surface on which the load is given, $S_2$ is the part of the surface with given displacements, and $t$ is the time. Here $P^i$ is understood to be the vector of the nonconservative external force. The meaning of the remaining quantities in (1. 1) is clarified as a result of the variation. The additional condition (1. 2) permits finding the value of the functional (1. 1) for given admissible functions and comparing its values corresponding to different admissible functions. The variational principle of dynamics written in such form can be used in deriving the fundamental relationships for a three-dimensional body under a nonconservative load [6].

For an independent variation of $u_i, \varepsilon_{ik}, \eta_{ik}, \theta^{ik}, \sigma^{ik}$ we can obtain from (1. 1) and (1. 2)

$$\int_V (E^{ikjl}\varepsilon_{jl} - \delta_{ik})\,\delta\varepsilon_{ik}dV - \tag{1.3}$$

$$\int_V \left[\theta^{ik} - \sigma^{ij}(\delta_j^k + \eta_{j\cdot}^{\cdot k})\right]\delta\eta_{ik}dV - \int_V \left(\overline{\nabla}_k\theta^{ik} - \rho\frac{\partial^2 u^i}{\partial t^2}\right)\delta u_i dV -$$

$$\int_V \left[\varepsilon_{ik} - \frac{1}{2}(\eta_{ik} + \eta_{ki} + \eta_{i\cdot}^{\cdot j}\eta_{kj})\right]\delta\sigma_{ik}dV - \int_V (\eta_{ik} - \overline{\nabla}_i u_k)\,\delta\theta^{ik}dV -$$

$$\int_{S_1} (\theta^{ik}n_i - P^k)\,\delta u_k dS - \int_{S_2} (u_k - U_k)\,n_i\delta\theta^{ik}dS = 0$$

from which follow the equations of motion, the kinematic relationships, the elasticity relationships, the natural boundary conditions for the stresses and displacements of a three-dimensional body under large displacements.

**2. Variational principle for a layer.** A shallow shell of alternating elastic isotropic layers of a reinforcing material and a binder (matrix) is considered. All the reinforcing layers are identical, the thickness of each layer is $h_A$ and the density of the material is $\rho_A$. All the binder layers are also identical, their thickness is denoted by $h_M$ and density by $\rho_M$. The thickness of the shell "as a whole" is $h$. If the number $n = h/(h_A + h_M)$ corresponding to the quantity of pairs of layers is an integer, the shell construction is nonsymmetric in thickness. In the case of a structure symmetric relative to the middle surface, the number $n$ is not an integer and depends on the relationship between the layer thicknesses.

It is possible to go from the functional for a three-dimensional body to the functional of shell theory for a layer by using the following hypotheses [7] (the reinforcing layer is used as an illustration without limiting generality):

the metric remains invariant over the layer thickness (and over the thickness of the whole shell); $a_{\alpha\beta}$ and $b_{\alpha\beta}$ $(\alpha, \beta = 1,2)$ are tensors of the first and second quadratic forms of the shell middle surface with which the curvilinear coordinates $x_A^1$, $x_A^2$, $x_A^3$ are related ($x_A^3$ is directed along the normal);

the quantities being varied change across the layer thickness in conformity with the dependences

$$u_{\alpha A} = v_{\alpha A} + x_A^3\psi_{\alpha A}, \quad u_{3A} = w_A + x_A^3\psi_{3A} \tag{2.1}$$

$$\sigma_A^{\alpha\beta} = \frac{1}{h_A}T_A^{\alpha\beta} + \frac{12x_A^3}{h_A^2}M_A^{\alpha\beta}, \quad \sigma_A^{\alpha 3} = \frac{1}{h_A}Q_A^{\alpha}, \quad \sigma_A^{33} = \frac{1}{h_A}R_A$$

$$\theta_A^{\alpha\beta} = \frac{1}{h_A} t_A^{\alpha\beta} + \frac{12x_A{}^3}{h_A{}^3} s_A^{\alpha\beta}, \quad \theta_A^{\alpha 3} = \frac{1}{h_A}\left(c_A^\alpha + \frac{12x_A{}^3}{h_A{}^3} d_A^\alpha\right)$$

$$\theta_A^{3\alpha} = \frac{1}{h_A} l_A^\alpha, \quad \theta_A^{33} = \frac{1}{h_A} l_A^3$$

$$\varepsilon_{\alpha\beta A} = \gamma_{\alpha\beta A} + x_A{}^3 k_{\alpha\beta A}, \quad \varepsilon_{\alpha 3A}(x_A{}^1, x_A{}^2, x_A{}^3, t) = \gamma_{\alpha 3A}(x_A{}^1, x_A{}^2, t)$$

$$\varepsilon_{33A} = \gamma_{33A}, \eta_{\alpha\beta A} = e_{\alpha\beta A} + x_A{}^3 \zeta_{\alpha\beta A}, \eta_{3\alpha A} = \iota_{\alpha A}, \quad \eta_{\alpha 3A} = \omega_{\alpha A} + x_A{}^3 \mu_{\alpha A}$$

$$\eta_{33A} = \iota_{3A}$$

There results from the relationships three to nine in (2.1) that

$$\{T_A^{\alpha\beta}, M_A^{\alpha\beta}, Q_A{}^\alpha, R_A\} = \int \{\sigma_A^{\alpha\beta}, x_A{}^3 \sigma_A^{\alpha\beta}, \sigma_A^{\alpha 3}, \sigma_A^{33}\}\, dx_A{}^3$$

$$\int \{\sigma_A^{\alpha 3}, \sigma_A^{33}\}\, x_A{}^3 dx_A{}^3 = 0$$

$$\{t_A^{\alpha\beta}, s_A^{\alpha\beta}, c_A^{\alpha\beta}, l_A^{\alpha\beta}, y_A{}^\alpha\} = \int \{\theta_A^{\alpha\beta}, x_A{}^3 \theta_A^{\alpha\beta}, \theta_A^{\alpha 3}, \theta_A^{3\alpha}, \theta_A^{33}\}\, dx_A{}^3$$

The integrals here are taken between the limits $-h_A/2$ and $h_A/2$.

The following notation

$$\{q_{\alpha A}, q_A, p_{\alpha A}, p_{3A}\} = \{P_{\alpha A}, P_{3A}, x_A{}^3 P_{\alpha A}, x_A{}^3 P_{3A}\}\Big|_{x_A{}^3 = -h_A/2}^{x_A{}^3 = h_A/2}$$

$$\{N_{\alpha A}, N_A, K_{\alpha A}\} = \int_{-h/2}^{h/2} \{P_{\alpha A}, P_{3A}, x_A{}^3 P_{\alpha A}\}\, dx_A{}^3$$

is introduced for the loads acting on the outer surfaces and along the layer outlines.

Substitution of the relationships presented into (1.1) and (1.2) and integration over the layer thickness permits obtaining an expression of the variational principle for the layer-shell $\delta J_{1A} = 0$. All the equations and dynamics relationships for an elastic isotropic reinforcing layer-shell, which are analogous to the equations in [7, 8], follow from this variational principle. Corresponding relations are derived also for the matrix layer.

**3. Variational principle for a multilayered shell.** The $x^1$, $x^2$, $z$ coordinate system is connected with the middle surface of a multilayered shell. It is assumed that the displacement vector components are distributed as follows over the shell thickness

$$u_\alpha = v_\alpha + z\varphi_\alpha, \quad u_3 = w + z\varphi_3 \tag{3.1}$$

The displacements $u_\alpha$ and $u_3$ agree, respectively, with $v_{\alpha A}$, $v_{\alpha M}$ and $w_A$, $w_M$ on the coordinate surfaces of the separate layers.

The condition that the displacements on the contact surfaces of adjacent layers is equal, reduces by analogy with [9] to the relationships

$$\varphi_\alpha = s\psi_{\alpha A} + (1 - s)\psi_{\alpha M}, \quad \varphi_3 = s\psi_{3A} + (1 - s)\psi_{3M} \tag{3.2}$$

Here $s = h_A/(h_A + h_M)$.

In describing an equivalent shell model from a quasi-homogeneous material, the quantities $\psi_{\alpha M}$ and $\psi_{3M}$ are eliminated from consideration by using the relationships (3.2); the subscript $A$ in the $\psi_{\alpha A}, \psi_{3A}$ and other quantities characterizing the reinforcing layers is omitted.

The following hypotheses about the strain distribution over the thickness of an equivalent continuous shell are introduced:

$$\gamma_{\alpha\beta A}=\gamma_{\alpha\beta M}=\gamma_{\alpha\beta}+z\varkappa_{\alpha\beta} \tag{3.3}$$
$$sk_{\alpha\beta A}+(1-s)\,k_{\alpha\beta M}=\varkappa_{\alpha\beta},\quad \vartheta_{\alpha\beta}=\varkappa_{\alpha\beta}-k_{\alpha\beta A}$$
$$s\gamma_{\alpha 3A}+(1-s)\,\gamma_{\alpha 3M}=f(z)\,\beta_\alpha,\quad (1-s)(\gamma_{\alpha 3M}-\gamma_{\alpha 3A})=f(z)\,\xi_\alpha$$
$$s\gamma_{33A}+(1-s)\,\gamma_{33M}=f^*(z)\,\beta_3,\quad (1-s)(\gamma_{33M}-\gamma_{33A})=f^*(z)\,\xi_3$$
$$e_{\alpha\beta A}=e_{\alpha\beta M}=e_{\alpha\beta}+z\nu_{\alpha\beta},\quad s\zeta_{\alpha\beta A}+(1-s)\,\zeta_{\alpha\beta M}=\nu_{\alpha\beta}$$
$$\omega_{\alpha A}=\omega_{\alpha M}=\omega_\alpha+z\pi_\alpha,\quad s\mu_{\alpha A}+(1-s)\,\mu_{\alpha M}=\pi_\alpha$$
$$sv_{\alpha A}+(1-s)\,v_{\alpha M}=v_\alpha,\quad sv_{3A}+(1-s)\,v_{3M}=v_3$$

The distribution of the internal forces and moments is assumed to be according to the following laws:

$$n\,(T_A^{\alpha\beta}+T_M^{\alpha\beta})=T^{\alpha\beta}+\frac{12z}{h^2}\,M^{\alpha\beta} \tag{3.4}$$
$$n\,(Q_A{}^\alpha+Q_M{}^\alpha)=f(z)\,Q^\alpha,\quad n\left(\frac{s}{1-s}\,Q_M{}^\alpha-Q_A{}^\beta\right)=f(z)\,\tau^\alpha$$
$$n\,(M_A^{\alpha\beta}+M_M^{\alpha\beta})=m^{\alpha\beta},\quad \frac{n}{1-s}\,M_M^{\alpha\beta}=\chi^{\alpha\beta}$$
$$n\,(R_A+R_M)=f^*(z)\,R,\quad n\left(\frac{s}{1-s}\,R_M-R_A\right)=f^*(z)\,r$$
$$n\,(t_A^{\alpha\beta}+t_M^{\alpha\beta})=t^{\alpha\beta}+\frac{12z}{h^2}\,y^{\alpha\beta},\quad n\,(s_A^{\alpha\beta}+s_M^{\alpha\beta})=$$
$$s^{\alpha\beta},\quad \frac{n}{1-s}\,s_M^{\alpha\beta}=z^{\alpha\beta}$$
$$n\,(c_A{}^\alpha+c_M{}^\alpha)=f(z)\,(c^\alpha+z\vartheta^\alpha)$$
$$n\,(d_A{}^\alpha+d_M{}^\alpha)=f(z)\,d^\alpha,\quad \frac{n}{1-s}\,d_M{}^\beta=f(z)\,\alpha^\beta$$
$$n\,(j_A^\alpha+j_M{}^\alpha)=f(z)\,j^\alpha,\quad \frac{n}{1-s}\,j_M{}^\alpha=f(z)\,(\rho^\alpha+j^\alpha)$$
$$n\,(j_A{}^3+j_M{}^3)=f^*(z)\,j^3,\quad \frac{n}{1-s}\,j_M{}^3=f^*(z)\,(\rho^3+j^3)$$

Therefore, the distribution of the longitudinal and flexural strains and microstrains, as well as of the corresponding forces and moments is taken to be linear over the thickness. The distribution of the transverse strains and forces over the thickness is described by using the even functions $f(z)$ and $f^*(z)$ which satisfy the conditions (here and everywhere below, the integrals are taken over the thickness of the shell as a whole unless specified otherwise)

$$\int f(z)\,\{1,z^2,f\}\,dz=h\left\{1,k_0,\frac{1}{k^2}\right\},\quad \int f^*(z)\,\{1,f^*\}\,dz=h\left\{1,\frac{1}{k^{*2}}\right\} \tag{3.5}$$

There follows from (3.4) and (3.5)

$$\{T^{\alpha\beta},M^{\alpha\beta},Q^\alpha\}=\frac{1}{h_A+h_M}\int\{T_A^{\alpha\beta}+T_M^{\alpha\beta},\,z\,(T_A^{\alpha\beta}+T_M^{\alpha\beta}),\,Q_A{}^\alpha+Q_M{}^\alpha\}\,dz \tag{3.6}$$
$$\{m^{\alpha\beta},\chi^{\alpha\beta},\tau^{\alpha\beta}\}=\frac{1}{h_A+h_M}\int\Big\{M_M^{\alpha\beta}+M_M^{\alpha\beta},\,\frac{1}{1-s}\,M_M^{\alpha\beta},$$
$$\left(\frac{s}{1-s}\,Q_M{}^\alpha-Q_A{}^\alpha\right)\Big\}\,dz$$
$$\{t^{\alpha\beta},y^{\alpha\beta},s^{\alpha\beta},c^\alpha\}=\frac{1}{h_A+h_M}\int\{t_A^{\alpha\beta}+t_M^{\alpha\beta},\quad z\,(t_A^{\alpha\beta}+t_M^{\alpha\beta}),$$
$$s_A^{\alpha\beta}+s_M^{\alpha\beta},\,c_A^\alpha+c_M^\alpha\}\,dz$$

$$\{z^{\alpha\beta}, \vartheta^{\alpha}, d^{\alpha}, \alpha^{\beta}, j^{\alpha}, \rho^{\alpha}, j^{3}, \rho^{3}\} = \frac{1}{h_A + h_M}\int\Big\{\frac{1}{1-s} s_M^{\alpha\beta},$$

$$z(c_A{}^{\alpha} + c_M{}^{\alpha}), d_A{}^{\alpha} + d_M{}^{\alpha}, \quad \frac{1}{1-s} d_M{}^{\beta}, i_A{}^{\alpha} + i_M{}^{\alpha},$$

$$\frac{1}{1-s} i_M{}^{\alpha}, i_A{}^{3} + i_M{}^{3}, \quad \frac{1}{1-s} i_A{}^{3}\Big\} dz$$

The following notation is also introduced:

$$\rho = s\rho_A + (1-s)\rho_M, \quad I_A = \frac{1}{12} h_A{}^2 s\rho_A, \quad I_M = \frac{1}{12} h_M{}^2 (1-s)\rho_M$$

$$\{q_\alpha, q, p_\alpha\} = \{P_\alpha, P_3, zP_\alpha\}\Big|_{z=-h/2}^{z=h/2}$$

$$N^{\alpha} = \frac{1}{h_A + h_M}\int(N_A{}^{\alpha} + N_M{}^{\alpha})\,dz, \quad N = \frac{1}{h_A + h_M}\int(N_A + N_M)\,dz$$

$$K^{\alpha} = \frac{1}{h_A + h_M}\int\Big[z(N_A{}^{\alpha} + N_M{}^{\alpha}) + \frac{1}{1-s} K_M{}^{\alpha}\Big] dz$$

$$L^{\alpha} = \frac{1}{h_A + h_M}\int\Big(K_A{}^{\alpha} + \frac{s}{1-s} K_M{}^{\alpha}\Big) dz$$

The energy smoothing operation [10] applied to variational relationships with additional conditions for the reinforcing layer and the matrix layer results in formulation of a variational principle analogous to (1.1) with additional conditions of the type (1.2) for a multilayered shell considered as a continuous shell with internal moments.

The variational principle permits derivation of fundamental relations for an elastic multilayered shell subjected to nonconservative loads. These relations are written below.

The equations of motion are:

$$\nabla_\alpha t^{\alpha\beta} - b_\alpha{}^{\beta} c^{\alpha} - \rho h\frac{\partial^2 v^{\beta}}{\partial t^2} + q^{\beta} = 0, \quad \nabla_\alpha c^{\alpha} + b_{\alpha\beta} t^{\alpha\beta} - \rho h\frac{\partial^2 w}{\partial t^2} + q = 0$$

$$\nabla_\alpha (y^{\alpha\beta} + z^{\alpha\beta}) - b_\alpha{}^{\beta}(k_0\vartheta^{\alpha} + \alpha^{\alpha}) - (\rho^{\beta} + j^{\beta}) -$$

$$A_1\frac{\partial^2\varphi^{\beta}}{\partial t^2} + A_2\frac{\partial^2\psi^{\beta}}{\partial t^2} + p^{\alpha} = 0$$

$$\nabla_\alpha (s^{\alpha\beta} - z^{\alpha\beta}) - b_\alpha{}^{\beta}(d^{\alpha} - \alpha^{\alpha}) + \rho^{\beta} + A_2\frac{\partial^2\varphi^{\beta}}{\partial t^2} - A_3\frac{\partial^2\psi^{\beta}}{\partial t^2} = 0$$

$$b_{\alpha\beta}(y^{\alpha\beta} + z^{\alpha\beta}) + (j^3 + \rho^3) + p^3 = 0$$

$$b_{\alpha\beta}(s^{\alpha\beta} - z^{\alpha\beta}) + \rho^3 = 0$$

where

$$A_1 = \frac{1}{12}\rho h^3 + h\frac{I_M}{(1-s)^2}, \quad A_2 = hI_M\frac{s}{(1-s)^2}$$

$$A_3 = h\Big[I_A + I_M\frac{s^2}{(1-s)^2}\Big]$$

The kinematic relationships are:

$$e_{\alpha\beta} = \nabla_\alpha v_\beta - b_{\alpha\beta} w, \quad \nu_{\alpha\beta} = \nabla_\alpha \varphi_\beta - b_{\alpha\beta}\varphi_3$$

$$\omega_\alpha = \nabla_\alpha w + b_{\alpha\beta} v_\beta, \quad \pi_\alpha = \nabla_\alpha \varphi_3 + b_{\alpha\beta}\varphi^{\beta}$$

$$\zeta_{\alpha\beta} = \nabla_\alpha \psi_\beta - b_{\alpha\beta}\psi_3, \quad \mu_\alpha = \nabla_\alpha \psi_3 + b_{\alpha\beta}\psi^{\beta}$$

$$\upsilon_\alpha = \varphi_\alpha, \quad \iota_\alpha = \psi_\alpha, \quad \upsilon_3 = \varphi_3, \quad \iota_3 = \psi_3$$

$$\varkappa_{\alpha\beta} = {}^1/_2\,(\nu_{\alpha\beta} + \nu_{\beta\alpha} + e_{\alpha\cdot}^{\cdot\gamma}\nu_{\beta\gamma} + e_{\beta\cdot}^{\cdot\gamma}\nu_{\alpha\gamma} + \omega_\alpha\pi_\beta + \omega_\beta\pi_\alpha)$$
$$k_{\alpha\beta} = {}^1/_2\,(\zeta_{\alpha\beta} + \zeta_{\beta\alpha} + e_{\alpha\cdot}^{\cdot\gamma}\zeta_{\beta\gamma} + e_{\cdot\beta}^{\cdot\gamma}\zeta_{\alpha\gamma} + \omega_\alpha\mu_\beta + \omega_\beta\mu_\alpha)$$
$$\vartheta_{\alpha\beta} = \varkappa_{\alpha\beta} - k_{\alpha\beta}$$
$$\beta_\alpha = {}^1/_2k^2\,(\omega_\alpha + v_\alpha + e_{\alpha\beta}v^\beta),\quad \xi_\alpha = {}^1/_2k^2\,[v_\alpha - \iota_\alpha + e_{\alpha\beta}\,(v^\beta - \iota^\beta)]$$
$$\beta_3 = k^{*2}\,\frac{1}{2}\left[2v_3 + v^\alpha v_\alpha + \frac{1}{1-s}\,(v^\alpha - \iota^\alpha)\,(v_\alpha - \iota_\alpha)\right]$$
$$\xi_3 = k^{*2}\,\frac{1}{2}\left[2v_3 - 2\iota_3 + v^\alpha v_\alpha - \iota^\alpha\iota_\alpha + \frac{s}{1-s}\,(v^\alpha - \iota^\alpha)\,(v_\alpha - \iota_\alpha)\right]$$

The elasticity relationships are:

$$T^{\alpha\beta} = h\,[sE_A^{\alpha\beta\gamma\delta} + (1-s)\,E_M^{\alpha\beta\gamma\delta}]\,\gamma_{\gamma\delta} +$$
$$+ h\,[s\lambda_A + (1-s)\,\lambda_M]\,a^{\alpha\beta}\beta_3 + hs\,(\lambda_M - \lambda_A)\,a^{\alpha\beta}\xi_3$$
$$M^{\alpha\beta} = \frac{h^3}{12}\,[sE_A^{\alpha\beta\gamma\delta} + (1-s)\,E_M^{\alpha\beta\gamma\delta}]\,\varkappa_{\gamma\delta}$$
$$m^{\alpha\beta} = h\left[s\,\frac{h_A^2}{12}\,E_A^{\alpha\beta\gamma\delta} + (1-s)\,\frac{h_M^2}{12}\,E_M^{\alpha\beta\gamma\delta}\right]k_{\gamma\delta} + h\,\frac{h_M^2}{12}\,E_M^{\alpha\beta\gamma\delta}\vartheta_{\gamma\delta}$$
$$\chi^{\alpha\beta} = h\,\frac{h_M^2}{12}\,E_M^{\alpha\beta\gamma\delta}k_{\gamma\delta} + h\,\frac{h_M^2}{12\,(1-s)}\,E_M^{\alpha\beta\gamma\delta}\vartheta_{\gamma\delta}$$
$$Q^\alpha = 2h\,[s\mu_A + (1-s)\,\mu_M]\,\beta^\alpha + 2hs\,[\mu_M - \mu_A]\,\xi^\alpha$$
$$\tau^\alpha = 2hs\,(\mu_M - \mu_A)\,\beta^\alpha + 2hs\left(\mu_A + \frac{s}{1-s}\,\mu_M\right)\xi^\alpha$$
$$R = h\,[s\,(\lambda_A + 2\mu_A) + (1-s)\,(\lambda_M + 2\mu_M)]\,\beta_3 + hs\,(\lambda_M + 2\mu_M -$$
$$\lambda_A - 2\mu_A)\,\xi_3 + h\,[s\lambda_A + (1-s)\,\lambda_M]\,a^{\alpha\beta}\gamma_{\alpha\beta}$$
$$r = hs\left[\lambda_A + 2\mu_A + \frac{s}{1-s}\,(\lambda_M + 2\mu_M)\right]\xi_3 + hs\,(\lambda_M + 2\mu_M -$$
$$\lambda_A - 2\mu_A)\,\beta_3 + hs\,(\lambda_M - \lambda_A)\,a^{\alpha\beta}\gamma_{\alpha\beta}$$
$$t^{\alpha\beta} = T^{\alpha\gamma}\,(\delta_\gamma^{\ \beta} + e_{\gamma\cdot}^{\ \cdot\beta}) + (M^{\alpha\gamma} + \chi^{\alpha\gamma})\,v_{\gamma\cdot}^{\ \cdot\beta} + (m^{\alpha\gamma} - \chi^{\alpha\gamma})\,\zeta_{\gamma\cdot}^{\ \cdot\beta} + (Q^\alpha + \tau^\alpha)\,v^\beta$$
$$y^{\alpha\beta} + z^{\alpha\beta} = (M^{\alpha\gamma} + \chi^{\alpha\gamma})\,(\delta_\gamma^{\ \beta} + e_{\gamma\cdot}^{\ \cdot\beta})$$
$$s^{\alpha\beta} - z^{\alpha\beta} = (m^{\alpha\gamma} - \chi^{\alpha\gamma})\,(\delta_\gamma^{\ \beta} + e_{\gamma\cdot}^{\ \cdot\beta})$$
$$c^\alpha = Q^\alpha + (M^{\alpha\beta} + \chi^{\alpha\beta})\,\pi_\beta + (m^{\alpha\beta} - \chi^{\alpha\beta})\,\mu_\beta + T^{\alpha\beta}\,\omega_\beta$$
$$k_{03}^{\ \alpha} + \alpha^\alpha = (M^{\alpha\beta} - \chi^{\alpha\beta})\,\omega_\beta,\quad d^\alpha - \alpha^\alpha = (m^{\alpha\beta} - \chi^{\alpha\beta})\,\omega_\beta$$
$$j^\alpha + \rho^\alpha = (Q^\beta + \tau^\beta)\,(\delta_\beta^{\ \alpha} + e_{\cdot\beta}^{\ \alpha\cdot}) + (R + r)\left[v^\alpha + \frac{s}{1-s}\,(v^\alpha - \iota^\alpha)\right]$$
$$\rho^\alpha = \tau^\alpha\,(\delta_\beta^{\ \alpha} + e_{\cdot\beta}^{\ \alpha\cdot}) + r\left[\iota^\alpha + \frac{s}{1-s}\,(v^\alpha - \iota^\alpha)\right],\quad j^3 = R,\quad \rho^3 = r$$

The relationships

$$E_{A,M}^{\alpha\beta\gamma\delta} = \lambda_{A,M}a^{\alpha\beta}a^{\gamma\delta} + 2\mu_{A,M}a^{\alpha\gamma}a^{\beta\delta}$$

are used here.

The boundary conditions are:

on the contour $C_1$

$$t^{\alpha\beta}n_\alpha = N^\alpha,\quad c^\alpha n_\alpha = N,\quad y^{\alpha\beta} + z^{\alpha\beta} = K^\alpha,\quad s^{\alpha\beta} - z^{\alpha\beta} = L^\alpha$$

on the contour $C_2$

$$v_\alpha = V_\alpha,\quad w = W,\quad \varphi_\alpha = \Phi_\alpha,\quad \psi_\alpha = \Psi_\alpha$$

The equations and relationships obtained above result in known dependences in a number of particular cases [2—5, 9, 10].

## REFERENCES

1. Vol'mir, A. S., Nonlinear Dynamics of Plates and Shells. "Nauka", Moscow, 1972.
2. Grigoliuk, E. I. and Chulkov, P. P., Nonlinear equations of shallow multilayered shells of regular construction. Inzh. Zh., Mekhan. Tverd. Tela, № 1, 1967.
3. Prusakov, A. P. and Rasteriaev, Iu. K., Bending, stability and vibrations of multilayered plates of nonsymmetric construction. Trudy VII All-Union Conference on the Theory of Plates and Shells. "Nauka", Moscow, 1970.
4. Novichkov, Iu. N., Nonlinear theory and stability of thick sandwich shells. PMM Vol. 37, № 2, 1973.
5. Gershtein, M. S., Geometrically nonlinear equations of motion of an elastic multilayered shell. Zh. Mekhanika Polimerov, № 5, 1973.
6. Levinson, M., Application of the Galerkin and Ritz methods to nonconservative problems of elastic stability. Z. angew. Math. und Phys., Vol. 17, 1966.
7. Ainola L. Ia., Nonlinear theory of Timoshenko type for elastic shells. Izv. Akad. Nauk ESSR, Ser. Fiz.-Matem. i Tekhn. Nauk, Vol. 14 № 3, 1965.
8. Ainola, L. Ia., Variational methods for nonlinear equations of motion of shells. PMM Vol. 32, № 1, 1968.
9. Achenbach, J. D. and Herrmann G., Effective stiffness theory for a laminated composite. In: Proc. of 10th Midwest Mech. Conf., Boulder, Colo, 1968.
10. Bolotin, V. V., On the theory of laminar plates. Izv. Akad. Nauk SSSR, OTN, Mekhan. i Mashinostr., № 3, 1963.

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# ON THE ASYMPTOTICS OF UNSTEADY MOTION OF GAS SUBJECTED TO A MOMENTUM

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The motion of gas which initially fills the whole space and is subjected to an instantaneous liberation in a thin layer of initial internal energy $E_0$ and momentum $I_0$ is considered. The asymptotic behavior of solution for various relations between $E_0$ and $I_0$ is investigated numerically.

When solving unsteady problems of gasdynamics it is often interesting to investigate the asymptotic properties of motion, which are determined for a fairly long time $t$ and are independent of initial data details. In the majority of cases these properties are defined by self-similar solutions. The transition of the flow to the self-similar mode can be traced by solving the exact problem with initial and boundary conditions for the input Euler equations.

Let us consider the plane motion of a perfect inviscid gas free of thermal conductivity